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ORIGINAL PAPER

Dynamical behaviors of the brusselator system with impulsive input

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Abstract Responses of dynamic system to pulse perturbations were investigated theoretically and experimentally. The model used in this paper has been proved dissipative by impulsive and dynamic theory. Complex phenomena such as limit cycles, periodic solutions, and chaos were numerically demonstrated.

Keywords Brusselator system · Impulse · Dissipative · Chaos

1 Introduction

Most chemical reactions can present rich phenomena in vessels, such as chemical oscillations [1–5], period doubling, chemical waves [6,7], and chaos [8,9]. Analysis of forced nonlinear oscillations plays an important role in understanding their dynamic phenomena of electronic generators, mechanical, chemical and biological systems. Even small external disturbances are likely to change behaviors of dynamical systems. It is well known that Prigogine had put forward a model—the brusselator system, which was very simple in mathematics and but presented many complex phenomena in theory [10,11]. A lot of theoretical and experimental experts had reported in the past that the steady state was unstable and a limit cycle would

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appear under certain condition [11,12]. In 1952, Turing observed the brusselator system with a diffusion effect and noticed the "Turing Bifurcation" for the first time [13,14]. This explained a limit cycle could lead to chemical waves if diffusions were taken into accounted.

There exist more complicated phenomena [4,15,16] if the brusselator system goes on in open vessels. Actually, when experiments were done in laboratory, reactants may be injected into a vessel by means of constant input [17], periodic input [15] or impulsive input. To the best of our knowledge, however, few researchers focus on how the impulsive disturbance will influence the dynamic behaviors of brusselator system. For this reason we construct the brusselator system with impulse input and find some interesting phenomena for instance limit cycles, periodic solutions and chaos. In this paper, according to impulsive theory [18,19] and dynamical theory [12,20], we theoretically proved the model dissipation, numerically simulated bifurcations and chaos when the parameter varied within some regions.

The rest of this paper is structured as follows. In Sect. 2 we introduce the brusselator system with impulsive input. Section 3 is devoted to the system permanent by dynamic theory and impulsive theory. In Sect. 4 we conduct a short of bifurcation analysis and discuss the behaviors of chaos for both mathematical analysis and numerical simulations. In the last section we compare the brusselator system of constant input with that of impulsive input.

2 Brusselator system with impulsive input

Firstly, the brusselator system of substrate input can be shown as follows:

$$\begin{cases} \dot{x} = a - (b+1)x + x^2 y, \\ \dot{y} = bx - x^2 y. \end{cases} \tag{2.1}$$

We suppose an excessive amount of reactors A, B are designed into vessels, so their concentrations a, b do not change as time goes by; all rate constants are equal to 1; x(t) and y(t) are the concentration of both reactants with an initial concentration x_0 and y_0 (i.e. $x_0 > 0$, $y_0 > 0$), where $x, y \in R^+$, and $a, b \in R^+$. This model has been proved there exists a stable limit cycle (see Fig. 7) if the condition $b > 1 + a^2$ is satisfied [12].

With an impulsive effect, the above reaction becomes:

$$\begin{cases} \dot{x} = -(b+1)x + x^2 y, & t \neq n\tau, \\ \dot{y} = bx - x^2 y, & \\ x(t^+) = x(t) + a, & t = n\tau. \end{cases} \tag{2.2}$$

We assume the amount of reactor B is excessive, thus its concentration b does not change as time goes by; but, reactor A is ingested into vessels in amounts of a at the fixed time $n\tau$. Similarly, as a hypothesis all rate constants are equal to 1; x(t) and y(t) are the concentrations of both reactants with an initial concentration x_0 and y_0 respectively (i.e. $x_0 >$, $y_0 > 0$), where $x, y \in R^+$, $n \in N$, τ is an impulsive periodic time, and $a, b \in R^+$ are constants.

3 Permanent

If the parameter $a = 0$, Fig. 1 shows that the trajectory described by Eq. 2.2 finally falls on the y axis. The positive quadrant is divided into I, II, III regions by isoclinal lines of system (2.2).

Theorem 3.1 $R^2_{0+} = \{(x, y)|x \geq 0, y \geq 0\}$ *is an invariant domain of Eq. 2.2.*

Proof If $x_0 = 0$, then $\dot{x} = 0$, $\dot{y} = 0$, for all t $(0 < t < \tau)$. Due to the effect of impulsive disturbance, the initial point (x_0, y_0) jumps to point (a, y_0) at time $t = \tau$. We get

$$\frac{dy}{dt}|_{y=0} > 0, \quad at \quad x > 0.$$

The curves of Eq. 2.2 do not cross the x axis and enter the fourth quadrant. Therefore, if the initial point $(x_0, y_0) \in R^2_{+0}$, the cures described by Eq. 2.2 still belong to this region. This completes the proof of Theorem 3.1.

Introduce three different types of basic models, that are,

$$\begin{cases} \dot{u} = r_1 - r_2 u, \\ u(0) = u_0. \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dot{u} = -ru, & t \neq n\tau \\ u(t^+) = u(t) + p, & t = n\tau, \\ u(0^+) = u_0. \end{cases} \tag{3.2}$$

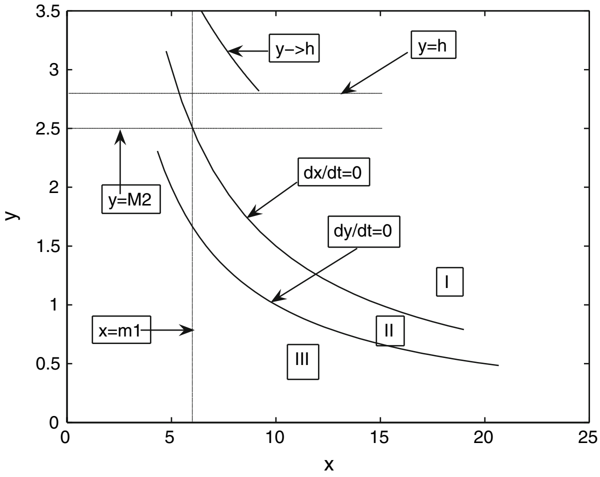

Fig. 1 $I = \{(x, y)|(\dot{x} < 0, \dot{y} > 0)\}$, $II = \{(x, y)|(\dot{x} < 0, \dot{y} < 0)\}$, $III = \{(x, y)|(\dot{x} > 0, \dot{y} < 0)\}$. The graph describes dynamical behaviors of system (2.2) at $a = 0$

and

$$\begin{cases} \dot{v}(t,x) \le g(t, v(t,x)), & t \neq n\tau \\ v(t, x(t^+)) \le \psi_n(v(t, x(t))), & t = n\tau. \end{cases} \tag{3.3}$$

□

For Eqs. 3.1, 3.2 and 3.3, we have the following conclusions.

Lemma 3.1 *Systems* (3.1) *have a positive equilibrium u^* and for every solution u of Eq. 3.1*

$$|u - u^*| \to 0, \quad as \quad t \to \infty,$$

where $u^* = \frac{r_1}{r_2}$.

Lemma 3.2 *Systems* (3.2) *have a positive periodic solution $u^*(t)$ and for every solution $u(t)$ of Eq. 3.2*

$$|u(t) - u^*(t)| \to 0, \quad as \quad t \to \infty,$$

where $u^*(t) = \frac{pe^{-(r(t-n\tau))}}{1-e^{-r\tau}}, t \in (n\tau, (n+1)\tau], n \in N$.

Lemma 3.3 *Let $v \in v_0$. Assume that systems (3. 3)*

$$\begin{cases} D^+V(t,x) \le g(t, V(t,x)), & t \neq n\tau, \\ V(t, x(t^+)) \le \psi_n(V(t, x(t))), & t = n\tau, \end{cases}$$

where g: $R_+ \times R_+ \longrightarrow R$ satisfies (H) and $\psi_n : R_+ \to R_+$ is non-decreasing.
(H) g is continuous in $(n\tau, (n+1)\tau] \times R$, and for $x \in R_+, n \in N$,
$\lim g(t, y) = g(n\tau^+, x)$ *exists, as* $(t, y) \to (n\tau^+, x)$.
Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} \dot{u} = g(t,u), & t \neq n\tau, \\ u(t^+) = \psi_n(u(t)), & t = n\tau, \\ u(0^+) = u_0. \end{cases}$$

Then $v(0^+, x_0) \le u_0$ implies that $v(t, x(t)) \le r(t)$, $t \ge 0$, where $x(t)$ is any solution of (2.2).

Theorem 3.2 *Systems* (2.2) *are permanent if* $(b+1)\tau > ln2$, *for* $0 < \tau < 1$, *or* $(b+1) > \ln 2$, *for* $\tau > 1$.

We will prove this conclusion through these following four lemmas.

Lemma 3.4 *For the solution $x(t)$ of Eq. 2.2, there exists a $T_1 > 0$, such that*

$$x(t) > m_1, as \quad t > T_1,$$

where m_1 is a positive constant.

Proof For all t since $y(t) \geq 0$, we get

$$\dot{x} \geq -(b+1)x.$$

By Lemma 3.2, we have

$$x(t) \geq u(t) \quad and \quad u(t) \to \overline{u}(t), \quad as \quad t \to \infty,$$

where u(t) is the solution of

$$\begin{cases} \dot{u} = -(b+1)u, & t \neq n\tau, \\ u(t^+) = u(t) + a, & t = n\tau, \\ u(0^+) = x_0, \end{cases}$$

and $\overline{u}(t) = \frac{ae^{-(b+1)(t-n\tau)}}{1-e^{-(b+1)\tau}}$, $t \in (n\tau, (n+1)\tau]$.

For $t \in (n\tau, (n+1)\tau]$, $\overline{u}(t)$ is a monotonic decreasing function of time t and has a minimum value m_1 at $t = (n+1)\tau$, that is,

$$m_1 = \frac{ae^{-(b+1)}}{1-e^{-(b+1)\tau}}.$$

So, there must exist a $T_1 > 0$, such that

$$x(t) \geq u(t) \geq \overline{u}(t) \geq m_1, \quad as \quad t > T_1.$$

Meanwhile, because the condition $m_1 < a$ is satisfied, there is no oscillation over the whole range of $0 < x(t) < a$. This completes the proof of Lemma 3.4. □

Lemma 3.5 *For the solution $y(t)$ of Eq. 2.2, there exists a $T_2 > T_1$, such that*

$$y(t) < M_2, \quad as \quad t > T_2,$$

where $M_2 = \frac{b+1}{m_1}$ is a positive constant.

Proof If $y_0 > M_2$, there must exist a $T_2 > T_1$, in order that

$$y(t) \leq M_2, \quad as \quad t \geq T_2.$$

Otherwise, by the trajectory in the I region, we assume a positive constant h satisfies the following formula

$$y(t) \to h \; and \quad x(t) \to \infty, \quad as \quad t \to \infty, \text{ (see Fig. 1)}$$

where $h > M_2$ is a positive constant, and $x(t)$ is the solution of Eq. 2.2.

Introduce a $V(t)$ function, that is,

$$V(t) = x(t) + y(t).$$

For any value of the parameter λ, we compute

$$\dot{V}(t) + \lambda V(t) = (\lambda - 1)x(t) + \lambda y(t),$$

choose $0 < \lambda_0 < 1$, such that

$$(\lambda_0 - 1)x(t) < 0 \; and \quad \dot{V}(t) + \lambda_0 V(t) < \lambda_0 y(t).$$

For $t \to \infty$, by the assumption

$$\lambda_0 y(t) \to \lambda_0 h.$$

Solve the following equations

$$\begin{cases} \dot{V}(t) + \lambda_0 V(t) < k, & t \neq n\tau, \\ V(t^+) = V(t) + a, & t = n\tau, \quad n \in N, t \to \infty, \\ V(0^+) = x_0 > 0. \end{cases}$$

By Lemma 3.3

$$V(t) \leq \left(V(0^+) - \tfrac{k}{\lambda_0}\right) e^{-(\lambda_0 t)} + \tfrac{a(1-e^{-(n\lambda_0\tau)})}{1-e^{(-\lambda_0\tau)}} + \tfrac{k}{\lambda_0}, t \in (n\tau, (n+1)\tau].$$

Therefore $V(t)$ is ultimately bounded, and it just contracts with the assumption

$$y(t) \to h, x(t) \to \infty, \; and \quad V(t) = x(t) + y(t) \to \infty, \; as \quad t \to \infty.$$

Hence, there exists a $T_2 > T_1$, such that

$$y(t) \leq M_2, \quad as \quad t \geq T_2.$$

If $y_0 < M_2$, in the I region we compute

$$\frac{dy}{dt} < 0, \quad for \; all \quad t > T_1.$$

So, whatever $y_0 \leq M_2 \; or \; y_0 > M_2$, there exists a $T_2 > T_1$, such that

$$y(t) \leq M_2, \quad as \quad t \geq T_2.$$

This completes the proof of Lemma 3.5. □

Lemma 3.6 *For the solution of Eq. 2.2, there exists a $T_3 > T_2$, such that*

$$x(t) \leq m_2, \quad as \quad t \geq T_3,$$

where $m_2 = \frac{k}{\lambda_0}$ is a positive constant.

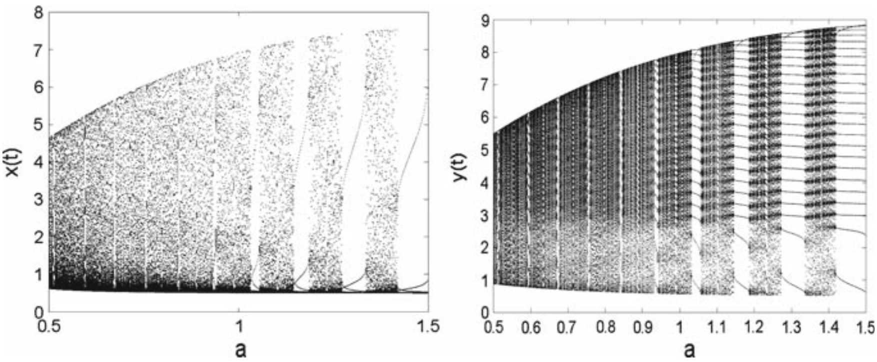

Fig. 2 The two graphs are that of bifurcations about the reactants of x(t) and y(t) respectively for $0.5 \leq a \leq 1.5$. Initial conditions : $(x_0, y_0) = (2, 1), b = 3.57, \tau = 1$

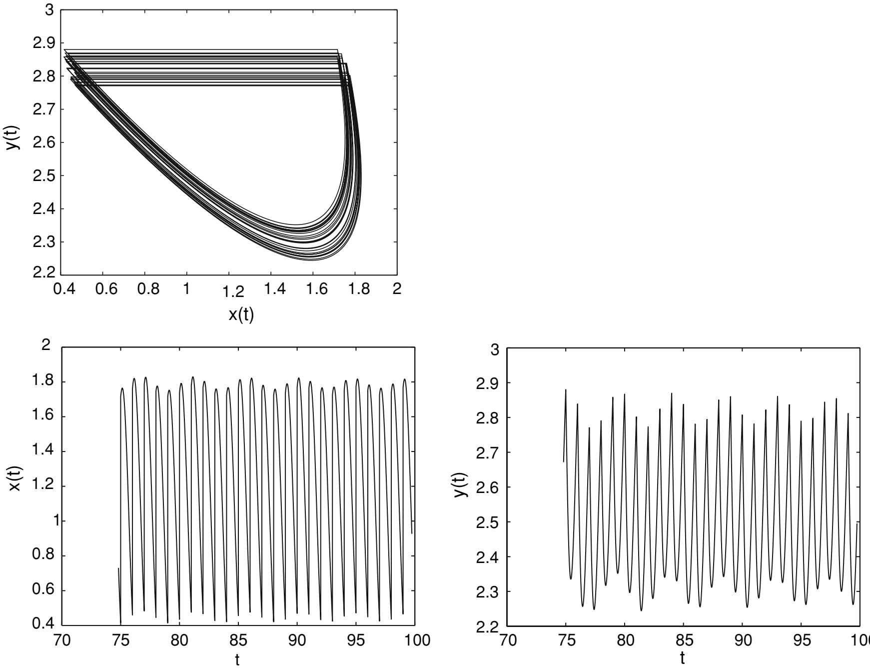

Fig. 3 The three graphs are those of reactants x–y plane and time series respectively. Initial conditions is the same with Fig. 2. At fixed a = 1.3 for $75 \leq t \leq 100$

Proof By means of Lemma 3.5, similar reasoning can judge the boundedness of $x(t)$ so long as $y(t)$ is bounded. Since

$$y(t) < M_2, \quad as \quad t > T_2,$$

we get

$$x(t) < \left(V(0^+) - \frac{k}{\lambda_0}\right) e^{-\lambda_0 t} + \frac{a(1-e^{-n\lambda_0 \tau})e^{-\lambda_0 (t-n\tau)}}{1-e^{-\lambda_0 \tau}} + \frac{k}{\lambda_0}, \quad as \quad t > T_2.$$

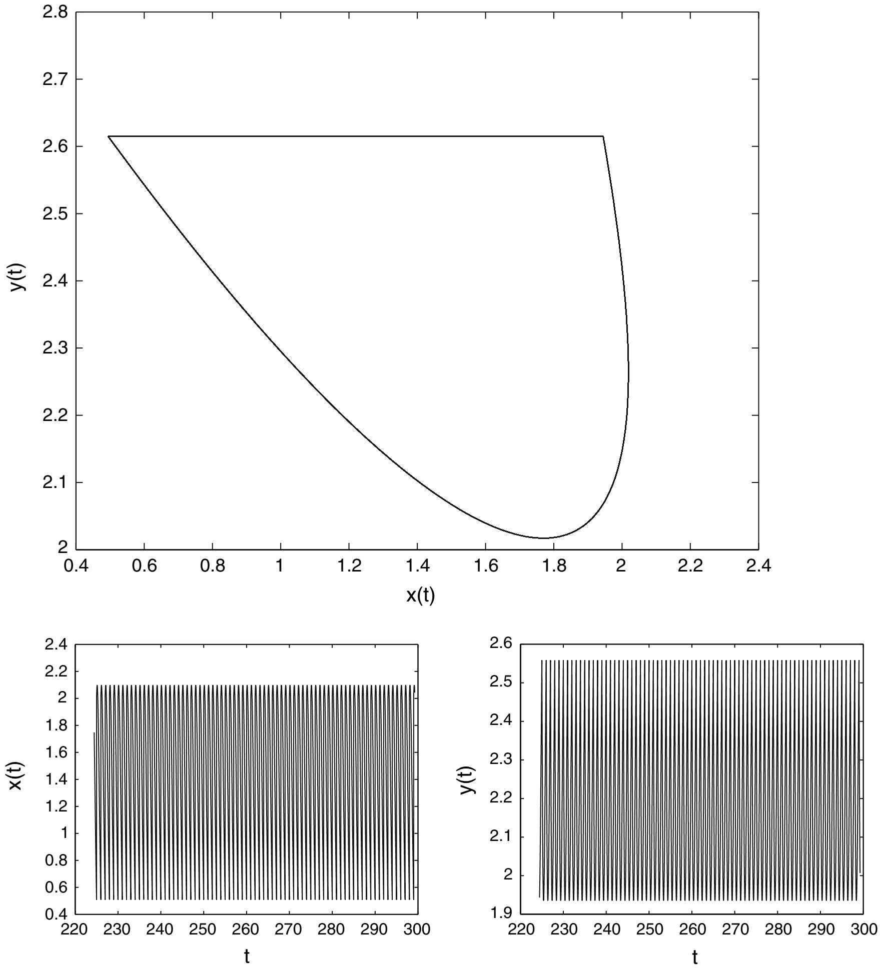

Fig. 4 Compare with Fig. 3. At the same a = 1.3, but for $225 \leq t \leq 300$. That shows Fig. 3 is convergent to a stable 1-periodic solution as $t \longrightarrow \infty$

Then, there exists a $T_3 > T_2$, such that

$$x(t) \leq m_2, \quad as \quad t > T_3.$$

This completes the proof of Lemma 3.6. □

Lemma 3.7 *For the solution $y(t)$ of Eq. 2.2, there exists a $T_4 > T_3$, such that*

$$y(t) > M_1, \quad as \quad t > T_4,$$

where M_1 is a positive constant.

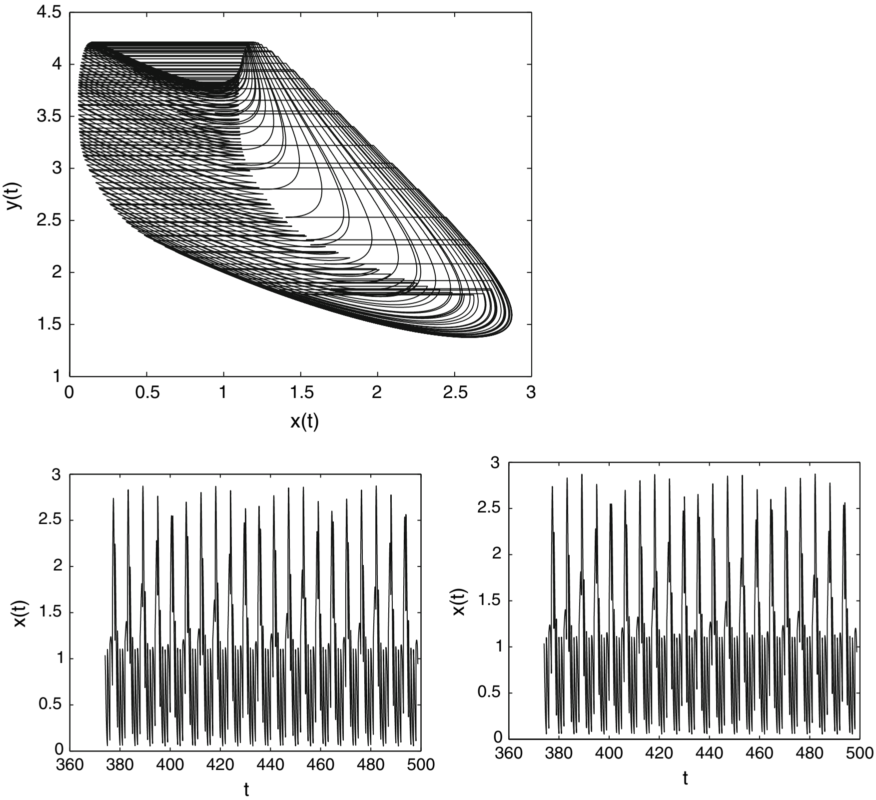

Fig. 5 Compare with Figs. 3 and 4, with the same initial conditions of Fig. 2, chaos occur for typical a = 1.16 in a shade region

Proof Since

$$m_1 \leq x(t) \leq m_2, \quad as \quad t > T_3,$$

we get

$$\dot{y} \geq bm_1 - m_2^2 y.$$

By Lemma 3.1, we have

$$y(t) \geq v^*, \quad as \quad t \to \infty,$$

where v^* is the solution of

$$\begin{cases} \dot{v} = bm_1 - m_2^2 v, \\ v(0) = y_0. \end{cases}$$

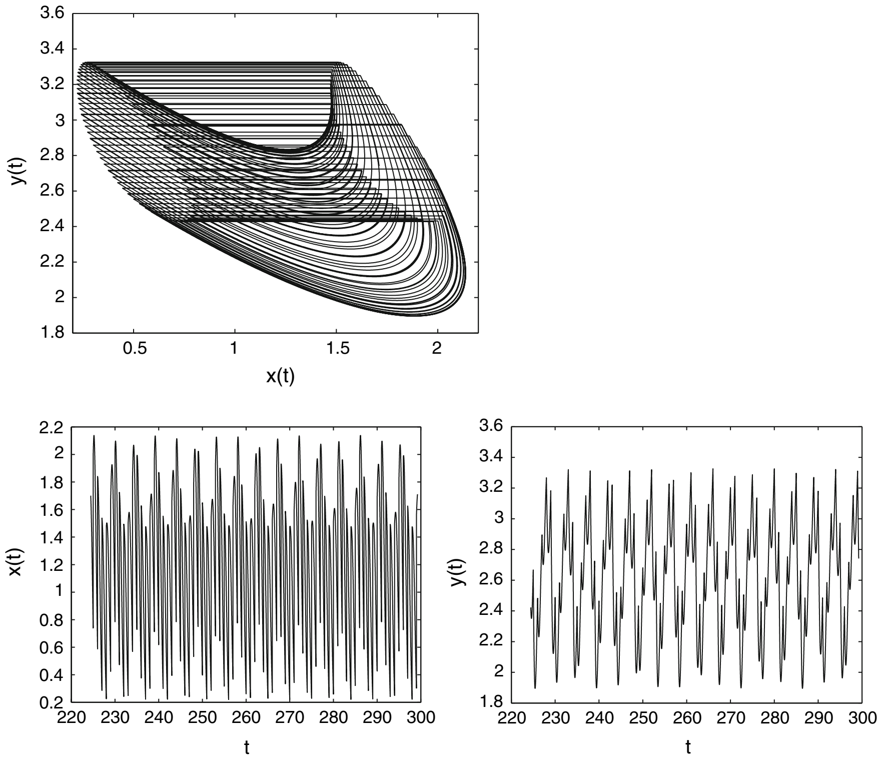

Fig. 6 Compare with Fig. 5 and with the same initial conditions of Fig. 5, a different behavior of chaos occurs for a = 1.25 in a different shade region

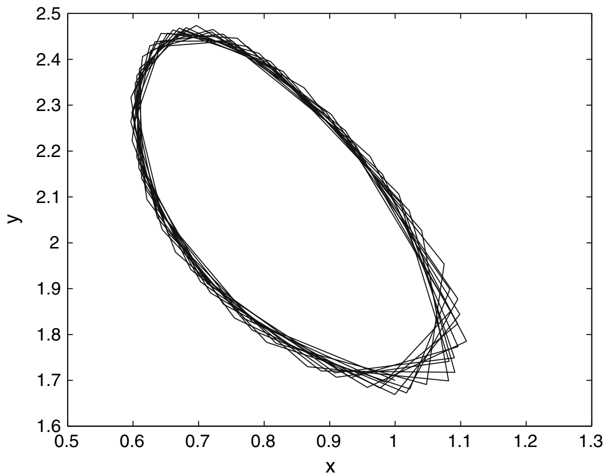

Fig. 7 The limit cycle of the brusselator system (2.1) with initial conditions $a = 0.8, b = 1.7$, $x_0 = 1$, $y_0 = 1.7$

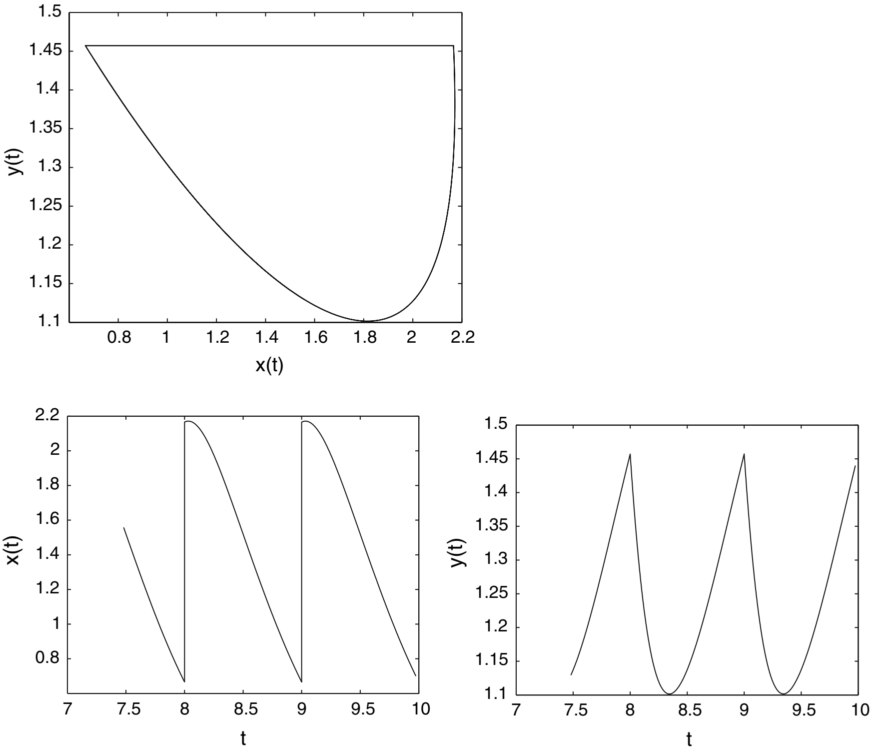

Fig. 8 Under such conditions $b < 1 + a^2$, the solution of system (2.2) is periodic solutions. These graphs are that of phase diagram and those of time series respectively with initial conditions $a = 1.5$, $b = 2$, $x_0 = 2$, $y_0 = 1.45$

So there exists a $T_4 > T_3$, such that

$$y(t) > M_1, \quad as \quad t > T_4,$$

where $M_1 = \frac{bm_1}{m_2}$ is a positive constant.
In the III region if the condition $y_0 > M_1$ is satisfied, we compute

$$\frac{dy}{dt} > 0, \quad for\ all \quad t > T_4.$$

Therefore, whatever $y_0 > M_1$ or $y_0 \leq M_1$, there exists a $T_4 > T_3$, such that

$$y(t) > M_1, \quad as \quad t > T_4.$$

This completes the proof of Lemma 3.7. □

By means of Lemmas 3.4–3.7, we complete the proof of Theorem 3.2.

4 Bifurcations and chaos

We think the bifurcation and chaos of the solution about system (2.2) as the change of the parameter a (i.e. the quality of the reactant A) and get the following conclusions.

Figure 2 shows, if the parameter a is within above-unshade regions, systems (2.2) is stable and convergent to a stable periodic solution, such as Figs. 3 and 4; but if the parameter a is within above-shade regions, systems (2.2) is unstable and chaos occur, such as Figs. 5 and 6. Therefore, these bifurcations reveal a transition from periodic solutions to chaos for systems (2.2) as the parameter a change in the region $0.5 \leq a \leq 1.5$.

5 Conclusions

Previous report [12] show that under such condition $b > 1 + a^2$ the brusselator system (2.1) has a stable limit cycle in the positive quadrant (see Fig. 7). Under the same condition, however, system (2.2) present periodic solutions and chaos by turns (see Figs. 2–6). Meanwhile under the contrast condition $b < 1+a^2$, we proved system (2.2) was a periodic solution (see Fig. 8).

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